fairly rapidly with N, reaches what appears to be a saturated state before actually beginning to fall with increasing N. At values of $N \geq 0.4$, the wall shear stress is reduced to levels below that of the stationary channel. The literature does not contain a reference to this unexpected behavior at high N, but this is not surprising because all of the previous calculations seem to be confined to $N \leq 0.15$.

Inspection of the shear- and rotation-production terms in the equation for $\overline{uv}$ provides a possible explanation for this behavior. These terms appear in that equation as

$$U_k \frac{\partial \overline{uv}}{\partial x_k} = -\overline{v^2}\frac{\partial U}{\partial y} - 2(\overline{u^2} - \overline{v^2})\Omega + \cdots \tag{6}$$

The quantity $(\overline{u^2} - \overline{v^2})$, although everywhere positive in a stationary channel, changes sign at high rates of rotation. Once this occurs, the rotation-production term becomes a sink in the $\overline{uv}$ equation, and this will act to reduce this quantity with increasing N. The DNS results support this interpretation: For $N = 0.5$, they show the difference in normal stresses to be negative over 63% of the channel cross section, with the levels of both the turbulence kinetic energy k and the structural parameter $(\overline{uv}/k)$ actually falling below their levels at $N = 0.15$. The causes of the exaggerated model behavior are not clear: It is possible that the pressure-strain models are not responding properly to the effects of high rotation or that the contribution of the pressure diffusion term can no longer be neglected in such conditions. Clearly, further work is needed to clarify this unexpected behavior.

Concluding Remarks

The applicability of linear and quadratic models for the fluctuating pressure-strain correlations is extended to flows with a system rotation by using the intrinsic spin tensor in place of the coordinate-dependent vorticity tensor. The extension is tested here for flow in a channel rotated about its spanwise axis but is equally applicable to all other models of rotation.

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Vortical Layer Analysis on Perturbed Conical Flow

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Nomenclature

a	= speed of sound
c_v	= specific heat at constant volume
G_{mn}	= shock-perturbation factor
K_δ	= hypersonic similarity parameter, $M_\infty \delta$
M_∞	= freestream Mach number
P	= pressure
S	= specific entropy
W	= azimuthal velocity perturbation
β	= semivertex angle of the unperturbed shock
γ	= ratio of specific heat (1.4 for air)
δ	= semivertex angle of the unperturbed cone
ε	= perturbation parameter (extremely small)
ρ	= density
σ	= shock relation, β/δ

Subscripts

m	= mth term in expansion
n	= nth term in expansion
0	= zeroth-order, unperturbed flow quantities

Introduction

REGARDING the development of the vortical layer problem, Ferri[1] proposed the vortical layer concept and analytically verified the invalidity of Stone's[2] results in an extremely thin vortical layer near the cone surface. This extremely small layer is required to adjust the entropy value from the outer solution to equalize the entropy value on the cone surface. Munsun[3] used the asymptotic matching principle and obtained uniformly valid solutions for the entire flowfield. He also found the azimuthal velocity and pressure in the outer expansion uniformly valid in the first-order perturbation. In 1985, Rasmussen[4] analyzed the shock layer in conical flow with transverse curvature. For inner expansion, he applied Munson's suggestion[3] of coordinate parameters and successfully obtained the analytical results for the vortical layer in the perturbed flow. Also, by using the matching method, the outer solutions and inner solutions are completely matched as the composite solutions, which are uniformly valid in the whole shock layer.

Recently, Lin and Rasmussen[5] combined the effects of longitudinal and transverse curvatures of conical flow and proceeded to carry out an outer perturbation expansion. Because the outer expansion solutions are not uniformly valid in the shock layer, however, the outcome near the conical body surface remains defective. To investigate the vortical layer near the cone surface for a conical body with multidirectional curvature, this study intends to discover uniformly

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valid analytical solutions in the shock layer by applying the inner perturbation expansions to analyze the characteristics in that region. Thereafter, the physical phenomenon inside the vortical layer can be investigated carefully.

Formulation of the Perturbed Conical Flow

The geometry of a perturbed conical body with multidirectional curvature is shown in Fig. 1. For convenience, this study adopts the spherical coordinates (r, θ, ϕ) to describe the flow. In the perturbed flow, the velocity vector is given by

$$\boldsymbol{V} = u\hat{e}_r + v\hat{e}_\theta + w\hat{e}_\phi \tag{1}$$

The basic conical body with combined longitudinal and transverse curvature can be expressed as

$$\theta_b = \delta\left[1 - \varepsilon_{mn}(r/l)^m \cos n\phi\right] \tag{2}$$

where l is the cone length. In addition, the relation of perturbed shock is

$$\theta_s = \delta\left[\sigma - \varepsilon_{mn}G_{mn}(r/l)^m \cos n\phi\right] \tag{3}$$

Because the perturbed cone shape is changed by the perturbation variables, the original spherical coordinate system is not valid in the perturbed cone and the perturbed shock. To deal with this, it is useful to introduce a new independent variable such that the boundary conditions are evaluated at the actual boundary location. Thus, we introduce a new stretched independent variable $\theta_0 = \theta_0(\theta, r, \phi)$ such that[5]

$$\frac{\theta_0 - \delta}{\beta - \delta} = \frac{\theta - \theta_b(r,\phi)}{\theta_s(r,\phi) - \theta_b(r,\phi)} \tag{4}$$

By assuming steady, inviscid, adiabatic, and perfect gas conditions, the governing equations of the mass, momentum, and energy must be transformed by the new independent variables for the outer expansion analysis.[5]

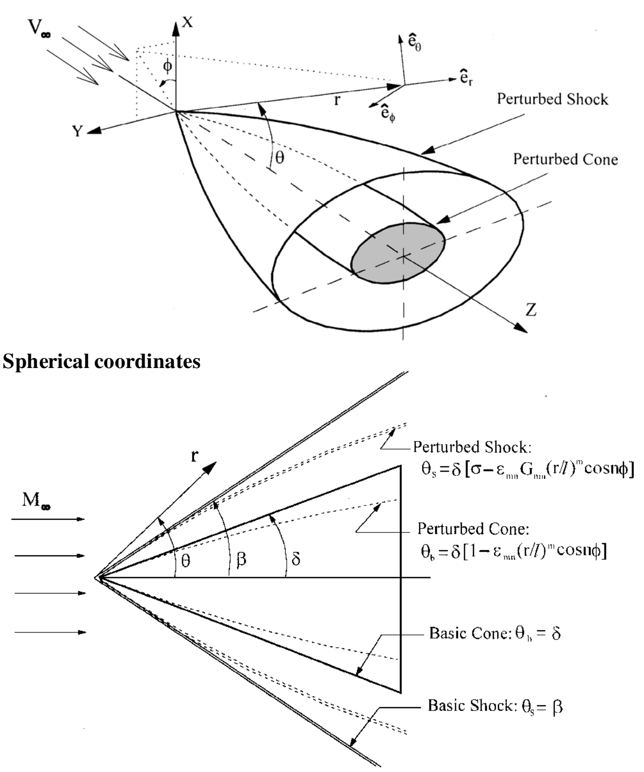

Fig. 1 Perturbed conical geometry.

First-Order Inner Perturbation Expansions

According to Munson's suggestion,[3] the inner coordinate variables for the vortical layer are introduced as

$$R = r, \qquad \Psi = \left[\frac{\theta - \theta_b(r,\phi)}{\theta_s - \theta_b(r,\phi)}\right]^{\varepsilon_{mn}} = \left[\frac{\theta_0 - \delta}{\beta - \delta}\right]^{\varepsilon_{mn}}, \qquad \Phi = \phi \tag{5}$$

We can rewrite the governing equations of the outer expansions in terms of the inner variables,[4] and thus the governing equations for the vortical layer are obtained.

The first-order inner perturbation expansions assume the following form[4]:

$$W^i(R,\Psi,\Phi;\varepsilon_{mn}) = \varepsilon_{mn}W^i_{mn}(R,\Psi,\Phi) + \mathcal{O}\left(\varepsilon^2_{mn}\right) \tag{6a}$$

$$P^i(R,\Psi,\Phi;\varepsilon_{mn}) = p_0(\delta)\left[1 + \varepsilon_{mn}P^i_{mn}(R,\Psi,\Phi) + \mathcal{O}\left(\varepsilon^2_{mn}\right)\right] \tag{6b}$$

$$\rho^i(R,\Psi,\Phi;\varepsilon_{mn}) = \rho_0(\delta)\left[1 + \varepsilon_{mn}\rho^i_{mn}(R,\Psi,\Phi) + \mathcal{O}\left(\varepsilon^2_{mn}\right)\right] \tag{6c}$$

$$S^i(R,\Psi,\Phi;\varepsilon_{mn}) = s_0 + c_v\left[\varepsilon_{mn}S^i_{mn}(R,\Psi,\Phi) + \mathcal{O}\left(\varepsilon^2_{mn}\right)\right] \tag{6d}$$

These expansions are to be substituted into the governing equations of the vortical layer, and the first-order perturbation equations are

$$\frac{\partial P^i_{mn}}{\partial \Psi} = 0 \tag{7a}$$

$$-2u_0(\delta)\Psi\frac{\partial S^i_{mn}}{\partial \Psi} + \frac{W^i_{mn}}{\delta}\frac{\partial S^i_{mn}}{\partial \Phi} = 0 \tag{7b}$$

$$S^i_{mn} = P^i_{mn} - \gamma\rho^i_{mn} \tag{7c}$$

When incorporated with the asymptotic matching principle,[6] the inner expansion of pressure is decided:

$$P^i(R,\Phi;\varepsilon_{mn}) = p_0(\delta)\left[1 + \varepsilon_{mn}P^o_{mn}(\delta)(R/l)^m\cos n\Phi + \mathcal{O}\left(\varepsilon^2_{mn}\right)\right] \tag{8}$$

The right-hand side of Eq. (8) is the outer expansion[5] evaluated at the cone surface. Therefore, the outer expansion for pressure is uniformly valid across the shock layer to all orders of ε_{mn}. Thus, the first-order inner pressure expressed in Eq. (7a) can be rewritten as

$$P^i_{mn}(R,\Phi) = P^o_{mn}(\delta)(R/l)^m\cos n\Phi \tag{9}$$

which is in accordance with Eq. (8). Moreover, with the preceding relations, the ϕ-momentum outer perturbation equation[5] is evaluated on the cone surface $\theta_0 = \delta$ to yield

$$P^o_{mn}(\delta) = \frac{(m+1)\gamma\delta u_0(\delta)}{na^2_0(\delta)}W^o_{mn}(\delta) \tag{10}$$

The first-order entropy can be obtained from Eq. (7b), which is a first-order partial differential equation having the characteristic equation

$$\mathrm{d}S^i_{mn} = 0 \tag{11}$$

Similarly, applying the asymptotic matching principle requires that the outer limit of the inner expansion ($\Psi \to 1$) equals the inner limit of the outer solution ($\theta_0 \to \delta$). When that is done, we obtain

$$S^i_{mn}(\Psi,\Phi) = S^o_{mn}(\beta)\left[\frac{\Psi^{A^*_{mn}} - \tan^2(n\Phi/2)}{\Psi^{A^*_{mn}} + \tan^2(n\Phi/2)}\right] \tag{12}$$

where $A^*_{mn} \equiv -[nW^o_{mn}(\delta)/\delta V_\infty]$. Obviously, after checking Eq. (12), S^i_{mn} approaches $S^o_{mn}(\beta)\cos n\Phi$ as $\Psi \to 1$, which is the behavior of the outer solution; $S^i_{mn} = -S^o_{mn}(\beta)$ on the cone surface is a constant value as $\Psi \to 0$, which is the correct behavior. Therefore, the entropy value calculated by Eq. (12) is uniformly valid across the shock layer. Meanwhile, the density of the inner solution is derived through Eq. (7c):

$$\rho^i_{mn}(R, \Psi, \Phi) = (1/\gamma)\left[P^o_{mn}(\delta)(R/l)^m \cos n\Phi - S^i_{mn}(\Psi, \Phi)\right] \quad (13)$$

Vortical Layer Analysis

With respect to the properties in the vortical layer, the pressure is verified to be uniformly valid for the whole shock layer from the preceding analysis. Figure 2 indicates the distributions of the first-order perturbed entropy for different values of Φ. It is normalized with the maximum value $(S^i_{mn})_{\max} = -S^o_{mn}(\beta)$. The entropy varies slowly across the shock layer toward the cone surface and demonstrates a rapid change until it approaches the outer edge of the vortical layer. In Fig. 2, all of the curves approach $S^i_{mn}(\theta_0)/(S^i_{mn})_{\max} = 1$ at the limit $\theta_0 \to \delta$.

In addition, when the flow approaches the conical body surface, it is obvious that the outer solution of density[5] is not valid on the conical body surface. Hence, by means of the additive method, we have the composite solution ρ^c_{mn} for the density at the base plane $r = l$:

$$\rho^c_{mn}(\theta_0, \phi) = \frac{1}{\gamma}\left\{P^o_{mn}(\theta_0)\cos n\phi - S^o_{mn}(\beta)\left[\frac{\Psi^{A^*_{mn}} - \tan^2(n\phi/2)}{\Psi^{A^*_{mn}} + \tan^2(n\phi/2)}\right]\right\} \quad (14)$$

The outer, inner, and composite density solutions for $\phi = 60$ deg are plotted and compared in Fig. 3 as a demonstration. As expected, the large variation near the cone surface cannot be predicted by the outer solution; also the inner solution loses its accuracy as θ_0 approaches

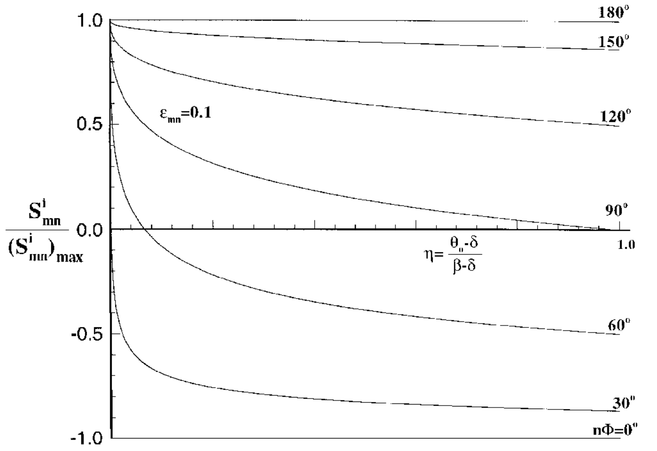

Fig. 2 First-order uniformly valid entropy variation in the shock layer for various azimuthal angles.

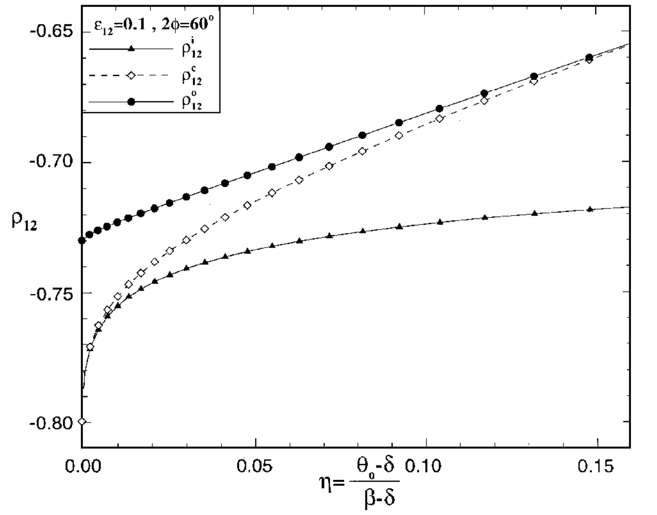

Fig. 3 First-order composite density solution from the matching of outer and inner solutions.

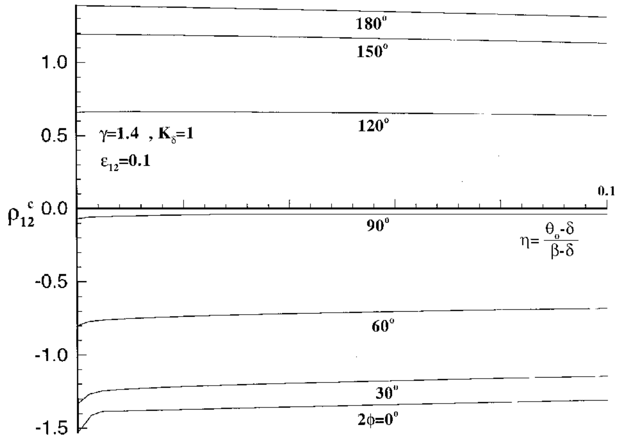

Fig. 4 First-order uniformly valid density variation near the cone surface for various azimuthal angles.

β. However, after asymptotically matching these two solutions, the composite solution is, thus, obtained and is uniformly valid across the whole shock layer. Similar density curves across the shock layer of ϕ are shown in Fig. 4 for different values. The conditions are for an elliptic cone with longitudinal curvature, $m = 1$ and $n = 2$. Note that only 10% of the shock layer close to the body is presented in Fig. 4 to show how very thin the vortical layer is.

Concluding Remarks

This study applies the perturbation method to move along the inner expansion to investigate the vortical layer associated with hypersonic flow past the perturbed cones. Large gradient changes in entropy and density are found when the flow approaches the cone surface. Also the pressure in the shock layer is shown to be uniformly valid, which agrees with previous studies. In addition, by setting $m = 0$ to remove the longitudinal perturbation, the present inner solution may be simplified to the inner solution of two-dimensional conical flow with transverse curvature, as accomplished previously. This study generalizes these results to take both longitudinal and transverse curvatures into account in a rigorous manner. To derive the expressions for properties inside the vortical layer, coordinate stretching is introduced to recast the governing equation describing the correct physical phenomena in this thin layer and can be used to move along the inner expansions. Therefore, incorporating the asymptotic matching principle into the inner and outer solutions, the uniformly valid approximation for the whole shock layer is obtained and expressed in closed form. The existence of the vortical layer implies that, even under inviscid conditions, the grids and numerics for numerical computation should be fashioned to accommodate this thin region of rapid adjustment.

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